

POLYNOMIALITY OF SOME HOOK-CONTENT SUMMATIONS FOR DOUBLED DISTINCT AND SELF-CONJUGATE PARTITIONS

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ABSTRACT. In 2009, the first author proved the Nekrasov-Okounkov formula on hook lengths for integer partitions by using an identity of Macdonald in the framework of type \tilde{A} affine root systems, and conjectured that some summations over the set of all partitions of size n are always polynomials in n . This conjecture was generalized and proved by Stanley. Recently, Pétréolle derived two Nekrasov-Okounkov type formulas for \tilde{C} and \tilde{C}^\sim which involve doubled distinct and self-conjugate partitions. Inspired by all those previous works, we establish the polynomiality of some hook-content summations for doubled distinct and self-conjugate partitions.

1. INTRODUCTION

The following so-called Nekrasov-Okounkov formula

$$\sum_{\lambda \in \mathcal{P}} q^{|\lambda|} \prod_{h \in \mathcal{H}(\lambda)} \left(1 - \frac{z}{h^2}\right) = \prod_{k \geq 1} (1 - q^k)^{z-1},$$

where \mathcal{P} is the set of all integer partitions λ with $|\lambda|$ denoting the size of λ and $\mathcal{H}(\lambda)$ the multiset of hook lengths associated with λ (see [6]), was discovered independently several times: First, by Nekrasov and Okounkov in their study of the theory of Seiberg-Witten on supersymmetric gauges in particle physics [16]; Then, proved by Westbury using D'Arcais polynomials [28]; Finally, by the first author using an identity of Macdonald [15] in the framework of type \tilde{A} affine root systems [6]. Moreover, he asked to find Nekrasov-Okounkov type formulas associated with other root systems [7, Problem 6.4], and conjectured that

$$n! \sum_{|\lambda|=n} \frac{1}{H(\lambda)^2} \sum_{h \in \mathcal{H}(\lambda)} h^{2k}$$

is always a polynomial in n for any $k \in \mathbb{N}$, where $H(\lambda) = \prod_{h \in \mathcal{H}(\lambda)} h$. This conjecture was proved by Stanley in a more general form. In particular, he showed that

$$n! \sum_{|\lambda|=n} \frac{1}{H(\lambda)^2} F_1(h^2 : h \in \mathcal{H}(\lambda)) F_2(c : c \in \mathcal{C}(\lambda))$$

is a polynomial in n for any symmetric functions F_1 and F_2 , where $\mathcal{C}(\lambda)$ is the multiset of contents associated with λ (see [24]). For some special functions F_1

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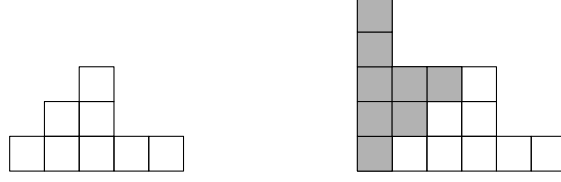


FIGURE 1. From strict partitions to doubled distinct partitions.

and F_2 the latter polynomial has explicit expression, as shown by Fujii, Kanno, Moriyama, Okada and Panova [4, 19].

A *strict partition* is a finite strict decreasing sequence of positive integers $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_\ell)$. The integer $|\bar{\lambda}| = \sum_{1 \leq i \leq \ell} \bar{\lambda}_i$ is called the *size* and $\ell(\bar{\lambda}) = \ell$ is called the *length* of $\bar{\lambda}$. For convenience, let $\bar{\lambda}_i = 0$ for $i > \ell(\bar{\lambda})$. A strict partition $\bar{\lambda}$ could be identical with its shifted Young diagram, which means that the i -th row of the usual Young diagram is shifted to the right by i boxes. We define the *doubled distinct partition* of $\bar{\lambda}$, denoted by $\bar{\lambda}\bar{\lambda}$, to be the usual partition whose Young diagram is obtained by adding $\bar{\lambda}_i$ boxes to the i -th column of the shifted Young diagram of $\bar{\lambda}$ for $1 \leq i \leq \ell(\bar{\lambda})$ (see [5, 20, 21]). For example, $(6, 4, 4, 1, 1)$ is the doubled distinct partition of $(5, 2, 1)$ (see Figure 1).

For each usual partition λ , let λ' denote the conjugate partition of λ (see [5, 15, 20, 21]). A usual partition λ is called *self-conjugate* if $\lambda = \lambda'$. The set of all doubled distinct partitions and the set of all self-conjugate partitions are denoted by \mathcal{DD} and \mathcal{SC} respectively. For each positive integer t , let

$$\mathcal{H}_t(\lambda) = \{h \in \mathcal{H}(\lambda) : h \equiv 0 \pmod{t}\}$$

be the multiset of hook lengths of multiples of t . Write $H_t(\lambda) = \prod_{h \in \mathcal{H}_t(\lambda)} h$.

Recently, Pétréolle derived two Nekrasov-Okounkov type formulas for \tilde{C} and \tilde{C}^* which involve doubled distinct and self-conjugate partitions. In particular, he obtained the following two formulas [20, 21].

Theorem 1.1 (Pétréolle [20, 21]). *For positive integers n and t we have*

$$(1.1) \quad \sum_{\substack{\lambda \in \mathcal{DD}, |\lambda|=2nt \\ \#\mathcal{H}_t(\lambda)=2n}} \frac{1}{H_t(\lambda)} = \frac{1}{(2t)^n n!}, \quad \text{if } t \text{ is odd};$$

$$(1.2) \quad \sum_{\substack{\lambda \in \mathcal{SC}, |\lambda|=2nt \\ \#\mathcal{H}_t(\lambda)=2n}} \frac{1}{H_t(\lambda)} = \frac{1}{(2t)^n n!}, \quad \text{if } t \text{ is even}.$$

Inspired by all those previous works, we establish the polynomiality of some hook-content summations for doubled distinct and self-conjugate partitions. Our main result is stated next.

Theorem 1.2. *Let t be a given positive integer. The following two summations for the positive integer n*

$$(1.3) \quad (2t)^n n! \sum_{\substack{\lambda \in \mathcal{DD}, |\lambda|=2nt \\ \#\mathcal{H}_t(\lambda)=2n}} \frac{F_1(h^2 : h \in \mathcal{H}(\lambda)) F_2(c : c \in \mathcal{C}(\lambda))}{H_t(\lambda)} \quad (t \text{ odd})$$

and

$$(1.4) \quad (2t)^n n! \sum_{\substack{\lambda \in \mathcal{SC}, |\lambda|=2nt \\ \#\mathcal{H}_t(\lambda)=2n}} \frac{F_1(h^2 : h \in \mathcal{H}(\lambda)) F_2(c : c \in \mathcal{C}(\lambda))}{H_t(\lambda)} \quad (t \text{ even})$$

are polynomials in n for any symmetric functions F_1 and F_2 .

In fact, the degrees of the two polynomials in Theorem 1.2 can be estimated explicitly in terms of F_1 and F_2 (see Corollary 4.8 and Theorem 5.3). When F_1 and F_2 are two constant symmetric functions, we derive Theorem 1.1. Other specializations are listed as follows.

Corollary 1.3. *We have*

$$(1.5) \quad (2t)^n n! \sum_{\substack{\lambda \in \mathcal{DD}, |\lambda|=2nt \\ \#\mathcal{H}_t(\lambda)=2n}} \frac{1}{H_t(\lambda)} \sum_{h \in \mathcal{H}(\lambda)} h^2 = 6t^2 n^2 + \frac{1}{3}(t^2 - 6t + 2)tn \quad (t \text{ odd}),$$

$$(1.6) \quad (2t)^n n! \sum_{\substack{\lambda \in \mathcal{SC}, |\lambda|=2nt \\ \#\mathcal{H}_t(\lambda)=2n}} \frac{1}{H_t(\lambda)} \sum_{h \in \mathcal{H}(\lambda)} h^2 = 6t^2 n^2 + \frac{1}{3}(t^2 - 6t - 1)tn \quad (t \text{ even}),$$

$$(1.7) \quad (2t)^n n! \sum_{\substack{\lambda \in \mathcal{DD}, |\lambda|=2nt \\ \#\mathcal{H}_t(\lambda)=2n}} \frac{1}{H_t(\lambda)} \sum_{c \in \mathcal{C}(\lambda)} c^2 = 2t^2 n^2 + \frac{1}{3}(t^2 - 6t + 2)tn \quad (t \text{ odd}),$$

$$(1.8) \quad (2t)^n n! \sum_{\substack{\lambda \in \mathcal{SC}, |\lambda|=2nt \\ \#\mathcal{H}_t(\lambda)=2n}} \frac{1}{H_t(\lambda)} \sum_{c \in \mathcal{C}(\lambda)} c^2 = 2t^2 n^2 + \frac{1}{3}(t^2 - 6t - 1)tn \quad (t \text{ even}).$$

The rest of the paper is essentially devoted to complete the proof of Theorem 1.2. The polynomiality of summations in (1.3) for $t = 1$ with $F_1 = 1$ or $F_2 = 1$ has an equivalent statement in terms of strict partitions, whose proof is given in Section 2. After recalling some basic definitions and properties of Littlewood decomposition in Section 3, the doubled distinct and self-conjugate cases of Theorem 1.2 are proved in Sections 4 and 5 respectively. Finally, Corollary 1.3 is proved in Section 6.

2. POLYNOMIALITY FOR STRICT AND DOUBLED DISTINCT PARTITIONS

In this section we prove an equivalent statement of the polynomiality of (1.3) for $t = 1$ with $F_1 = 1$ or $F_2 = 1$, which consists a summation over the set of strict partitions. Let $\bar{\lambda} = (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_\ell)$ be a strict partition. Therefore the leftmost box in the i -th row of the shifted Young diagram of $\bar{\lambda}$ has coordinate $(i, i + 1)$. The *hook length* of the (i, j) -box, denoted by $h_{(i,j)}$, is defined to be the number of boxes exactly to the right, or exactly above, or the box itself, plus $\bar{\lambda}_j$. For example, consider the box $\square = (i, j) = (1, 3)$ in the shifted Young diagram of the strict partition $(7, 5, 4, 1)$. There are 1 and 5 boxes below and to the right of the box \square respectively. Since $\bar{\lambda}_3 = 4$, the hook length of \square is equal to $1 + 5 + 1 + 4 = 11$, as illustrated in Figure 1. The *content* of $\square = (i, j)$ is defined to be $c_\square = j - i$, so that the leftmost box in each row has content 1. Also, let $\mathcal{H}(\bar{\lambda})$ be the multi-set of hook lengths of boxes and $H(\bar{\lambda})$ be the product of all hook lengths of boxes in $\bar{\lambda}$. The hook length and content multisets of the doubled distinct partition $\bar{\lambda}\bar{\lambda}$ can be obtained from $\mathcal{H}(\bar{\lambda})$ and $\mathcal{C}(\bar{\lambda})$ by the following relations:

$$(2.1) \quad \mathcal{H}(\bar{\lambda}\bar{\lambda}) = \mathcal{H}(\bar{\lambda}) \cup \mathcal{H}(\bar{\lambda}) \cup \{2\bar{\lambda}_1, 2\bar{\lambda}_2, \dots, 2\bar{\lambda}_\ell\} \setminus \{\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_\ell\},$$

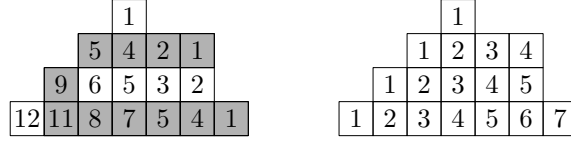


FIGURE 2. The shifted Young diagram, the hook lengths and the contents of the strict partition $(7, 5, 4, 1)$.

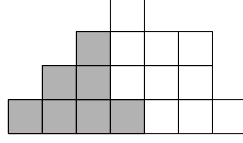


FIGURE 3. The skew shifted Young diagram of the skew strict partition $(7, 5, 4, 1)/(4, 2, 1)$.

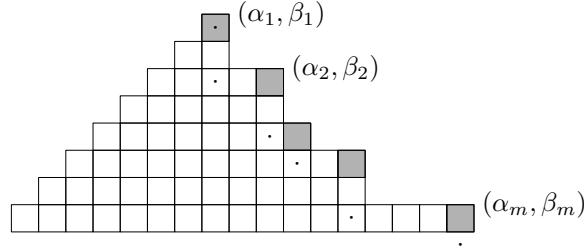


FIGURE 4. A strict partition and its corners. The outer corners are labelled with (α_i, β_i) ($i = 1, 2, \dots, m$). The inner corners are indicated by the dot symbol “.”.

$$(2.2) \quad \mathcal{C}(\bar{\lambda}\bar{\lambda}) = \mathcal{C}(\bar{\lambda}) \cup \{1 - c \mid c \in \mathcal{C}(\bar{\lambda})\}.$$

For two strict partitions $\bar{\lambda}$ and $\bar{\mu}$, we write $\bar{\lambda} \supseteq \bar{\mu}$ if $\bar{\lambda}_i \geq \bar{\mu}_i$ for any $i \geq 1$. In this case, the skew strict partition $\bar{\lambda}/\bar{\mu}$ is identical with the skew shifted Young diagram. For example, the skew strict partition $(7, 5, 4, 1)/(4, 2, 1)$ is represented by the white boxes in Figure 2. Let $f_{\bar{\lambda}}$ (resp. $f_{\bar{\lambda}/\bar{\mu}}$) be the number of standard shifted Young tableaux of shape $\bar{\lambda}$ (resp. $\bar{\lambda}/\bar{\mu}$). The following formulas for strict partitions are well-known (see [2, 23, 27]):

$$(2.3) \quad f_{\bar{\lambda}} = \frac{|\bar{\lambda}|!}{H(\bar{\lambda})} \quad \text{and} \quad \frac{1}{n!} \sum_{|\bar{\lambda}|=n} 2^{n-\ell(\bar{\lambda})} f_{\bar{\lambda}}^2 = 1.$$

Identity (1.1) with $t = 1$, obtained by Pétréolle, becomes

$$\sum_{\lambda \in \mathcal{D}, |\lambda|=2n} \frac{1}{H(\lambda)} = \frac{1}{2^n n!},$$

which is equivalent to the second identity of (2.3) in view of (2.1).

For a strict partition $\bar{\lambda}$, the *outer corners* (see [11]) are the boxes which can be removed in such a way that after removal the resulting diagram is still a shifted Young diagram of a strict partition. The coordinates of outer corners are denoted by $(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)$ such that $\alpha_1 > \alpha_2 > \dots > \alpha_m$. Let $y_j := \beta_j - \alpha_j$ ($1 \leq j \leq m$) be the contents of outer corners. We set $\alpha_{m+1} = 0$, $\beta_0 = \ell(\bar{\lambda}) + 1$ and call $(\alpha_1, \beta_0), (\alpha_2, \beta_1), \dots, (\alpha_{m+1}, \beta_m)$ the *inner corners* of $\bar{\lambda}$. Let $x_i = \beta_i - \alpha_{i+1}$ be the contents of inner corners for $0 \leq i \leq m$ (see Figure 3). The following relation of x_i and y_j are obvious.

$$(2.4) \quad x_0 = 1 \leq y_1 < x_1 < y_2 < x_2 < \dots < y_m < x_m.$$

Notice that $x_0 = y_1 = 1$ iff $\bar{\lambda}_{\ell(\bar{\lambda})} = 1$. Let $\bar{\lambda}^{i+} = \bar{\lambda} \cup \{\square_i\}$ such that $c_{\square_i} = x_i$ for $0 \leq i \leq m$. Here $\bar{\lambda}^{0+}$ does not exist if $y_1 = 1$. The set of contents of inner corners and the set of contents of outer corners of $\bar{\lambda}$ are denoted by $X(\bar{\lambda}) = \{x_0, x_1, \dots, x_m\}$ and $Y(\bar{\lambda}) = \{y_1, y_2, \dots, y_m\}$ respectively. The following relations between the hook lengths of $\bar{\lambda}$ and $\bar{\lambda}^{i+}$ are established in [11].

Theorem 2.1 (Theorem 3.1 of [11]). *Let $\bar{\lambda}$ be a strict partition with $X(\bar{\lambda}) = \{x_0, x_1, \dots, x_m\}$ and $Y(\bar{\lambda}) = \{y_1, y_2, \dots, y_m\}$. For $1 \leq i \leq m$, we have*

$$\begin{aligned} & \mathcal{H}(\bar{\lambda}) \cup \{1, x_i, 2x_i - 2\} \cup \{|x_i - x_j| : 1 \leq j \leq m, j \neq i\} \\ & \quad \cup \{x_i + x_j - 1 : 1 \leq j \leq m, j \neq i\} \\ & = \mathcal{H}(\bar{\lambda}^{i+}) \cup \{|x_i - y_j| : 1 \leq j \leq m\} \cup \{x_i + y_j - 1 : 1 \leq j \leq m\} \end{aligned}$$

and

$$\frac{H(\bar{\lambda})}{H(\bar{\lambda}^{i+})} = \frac{1}{2} \cdot \frac{\prod_{1 \leq j \leq m} \left(\binom{x_i}{2} - \binom{y_j}{2} \right)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} \left(\binom{x_i}{2} - \binom{x_j}{2} \right)}.$$

If $y_1 > 1$, we have

$$\begin{aligned} & \mathcal{H}(\bar{\lambda}) \cup \{1, x_1, x_1 - 1, x_2, x_2 - 1, \dots, x_m, x_m - 1\} \\ & = \mathcal{H}(\bar{\lambda}^{0+}) \cup \{y_1, y_1 - 1, y_2, y_2 - 1, \dots, y_m, y_m - 1\} \end{aligned}$$

and

$$\frac{H(\bar{\lambda})}{H(\bar{\lambda}^{0+})} = \frac{\prod_{1 \leq j \leq m} \left(\binom{x_0}{2} - \binom{y_j}{2} \right)}{\prod_{1 \leq j \leq m} \left(\binom{x_0}{2} - \binom{x_j}{2} \right)}.$$

Let k be a nonnegative integer, and $\nu = (\nu_1, \nu_2, \dots, \nu_{\ell(\nu)})$ be a usual partition. For arbitrary two finite alphabets A and B , the power sum of the alphabet $A - B$ is defined by [14, p.5]

$$(2.5) \quad \Psi^k(A, B) := \sum_{a \in A} a^k - \sum_{b \in B} b^k,$$

$$(2.6) \quad \Psi^\nu(A, B) := \prod_{j=1}^{\ell(\nu)} \Psi^{\nu_j}(A, B).$$

Let $\bar{\lambda}$ be a strict partition. We define

$$(2.7) \quad \Phi^\nu(\bar{\lambda}) := \Psi^\nu\left(\left\{\binom{x_i}{2}\right\}, \left\{\binom{y_i}{2}\right\}\right).$$

Theorem 2.2 (Theorem 3.5 of [11]). *Let k be a given nonnegative integer. Then, there exist some $\xi_j \in \mathbb{Q}$ such that*

$$\Phi^k(\bar{\lambda}^{i+}) - \Phi^k(\bar{\lambda}) = \sum_{j=0}^{k-1} \xi_j \binom{x_i}{2}^j$$

for every strict partition $\bar{\lambda}$ and $0 \leq i \leq m$, where x_0, x_1, \dots, x_m are the contents of inner corners of $\bar{\lambda}$.

Lemma 2.3. *Let k be a given nonnegative integer. Then, there exist some a_{ij} such that*

$$(x-y)^{2k} + (x+y-1)^{2k} = \sum_{i+j \leq k} a_{ij} \binom{x}{2}^i \binom{y}{2}^j$$

for every $x, y \in \mathbb{C}$.

Proof. The claim follows from

$$(x-y)^2 + (x+y-1)^2 = 4 \binom{x}{2} + 4 \binom{y}{2} + 1$$

and

$$(x-y)^2(x+y-1)^2 = \left(2 \binom{x}{2} - 2 \binom{y}{2}\right)^2. \quad \square$$

Lemma 2.4 (Theorem 3.2 of [11]). *Let k be a nonnegative integer. Then, there exist some $\xi_\nu \in \mathbb{Q}$ indexed by usual partitions ν such that*

$$\sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} (a_i - b_j)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} (a_i - a_j)} a_i^k = \sum_{|\nu| \leq k} \xi_\nu \Psi^\nu(\{a_i\}, \{b_i\})$$

for arbitrary complex numbers $a_0 < a_1 < \dots < a_m$ and $b_1 < b_2 < \dots < b_m$.

We define the difference operator \bar{D} for strict partitions by

$$(2.8) \quad \bar{D}(g(\bar{\lambda})) := 2 \sum_{i=1}^m g(\bar{\lambda}^{i+}) + g(\bar{\lambda}^{0+}) - g(\bar{\lambda}),$$

where $\bar{\lambda}$ is a strict partition and g is a function of strict partitions. In the above definition, the symbol $g(\bar{\lambda}^{0+})$ takes the value 0 if $\bar{\lambda}^{0+}$ does not exist, or equivalently if $\bar{\lambda}_{\ell(\bar{\lambda})} = 1$. By Theorem 2.1, we have

$$(2.9) \quad \bar{D}\left(\frac{1}{H(\bar{\lambda})}\right) = 0.$$

Theorem 2.5 (Theorem 2.3 of [11]). *Let g be a function of strict partitions and $\bar{\mu}$ be a given strict partition. Then we have*

$$(2.10) \quad \sum_{|\bar{\lambda}/\bar{\mu}|=n} 2^{|\bar{\lambda}|-|\bar{\mu}|-\ell(\bar{\lambda})+\ell(\bar{\mu})} f_{\bar{\lambda}/\bar{\mu}} g(\bar{\lambda}) = \sum_{k=0}^n \binom{n}{k} \bar{D}^k g(\bar{\mu})$$

and

$$(2.11) \quad \bar{D}^n g(\bar{\mu}) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} \sum_{|\bar{\lambda}/\bar{\mu}|=k} 2^{|\bar{\lambda}|-|\bar{\mu}|-\ell(\bar{\lambda})+\ell(\bar{\mu})} f_{\bar{\lambda}/\bar{\mu}} g(\bar{\lambda}).$$

In particular, if there exists some positive integer r such that $\bar{D}^r g(\bar{\lambda}) = 0$ for every strict partition $\bar{\lambda}$, then the left-hand side of (2.10) is a polynomial of n with degree at most $r - 1$.

For each usual partition δ let

$$p^\delta(\bar{\lambda}) := \Psi^\delta(\{h^2 : h \in \mathcal{H}(\bar{\lambda})\}, \emptyset).$$

By (2.1), we have

$$p^k(\bar{\lambda}) = \sum_{h \in \mathcal{H}(\bar{\lambda})} h^{2k} = 2 \sum_{h \in \mathcal{H}(\bar{\lambda})} h^{2k} + (4^k - 1) \sum_{i=1}^{\ell(\bar{\lambda})} \bar{\lambda}_i^{2k}$$

for a nonnegative integer k .

Theorem 2.6. Suppose that ν and δ are two given usual partitions. Then,

$$(2.12) \quad \bar{D}^r \left(\frac{p^\delta(\bar{\lambda}) \Phi^\nu(\bar{\lambda})}{H(\bar{\lambda})} \right) = 0$$

for every strict partition $\bar{\lambda}$, where $r = |\delta| + \ell(\delta) + |\nu| + 1$. Consequently, for a given strict partition $\bar{\mu}$,

$$(2.13) \quad \sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{|\bar{\lambda}|-\ell(\bar{\lambda})} f_{\bar{\lambda}/\bar{\mu}} p^\delta(\bar{\lambda})}{H(\bar{\lambda})}$$

is a polynomial in n of degree at most $|\delta| + \ell(\delta)$.

Proof. Let $X(\bar{\lambda}) = \{x_0, x_1, \dots, x_m\}$ and $Y(\bar{\lambda}) = \{y_1, y_2, \dots, y_m\}$. First, we show that the difference $p^k(\bar{\lambda}^{i+}) - p^k(\bar{\lambda})$ can be written as the following form

$$\sum_{j=0}^k \eta_j(\bar{\lambda}) \binom{x_i}{2}^j$$

for $0 \leq i \leq m$ and a nonnegative integer k , where each coefficient $\eta_j(\bar{\lambda})$ is a linear combination of some $\Phi^\tau(\bar{\lambda})$ for some usual partition τ of size $|\tau| \leq k$. Indeed, by Lemma 2.3 and Theorem 2.1,

$$\begin{aligned} p^k(\bar{\lambda}^{0+}) - p^k(\bar{\lambda}) &= 2 \sum_{j=1}^m (x_j^{2k} + (x_j - 1)^{2k}) - 2 \sum_{j=1}^m (y_j^{2k} + (y_j - 1)^{2k}) + 2^{2k} + 1 \\ &= \eta_0(\bar{\lambda}) = \sum_{j=0}^k \eta_j(\bar{\lambda}) \binom{x_0}{2}^j \quad [\text{ if } i = 0 \text{ and } \bar{\lambda}_{\ell(\bar{\lambda})} \geq 2] \end{aligned}$$

and

$$\begin{aligned} &p^k(\bar{\lambda}^{i+}) - p^k(\bar{\lambda}) \\ &= 2 \sum_{j=1}^m ((x_i - x_j)^{2k} + (x_i + x_j - 1)^{2k}) - 2 \sum_{j=1}^m ((x_i - y_j)^{2k} + (x_i + y_j - 1)^{2k}) \\ &\quad + 2x_i^{2k} + 2(2x_i - 2)^{2k} + 2 - 2(2x_i - 1)^{2k} + (2^{2k} - 1)(x_i^{2k} - (x_i - 1)^{2k}) \\ &= \sum_{j=0}^k \eta_j(\bar{\lambda}) \binom{x_i}{2}^j \quad [\text{ if } 1 \leq i \leq m]. \end{aligned}$$

Next, let $A = \Phi^\nu(\bar{\lambda})$ and $B = p^\delta(\bar{\lambda})$. We have

$$\begin{aligned}\Delta_i A &:= \Phi^\nu(\bar{\lambda}^{i+}) - \Phi^\nu(\bar{\lambda}) = \sum_{(*)} \prod_{s \in U} \Phi^{\nu_s}(\bar{\lambda}) \prod_{s' \in V} (\Phi^{\nu_{s'}}(\bar{\lambda}^{i+}) - \Phi^{\nu_{s'}}(\bar{\lambda})), \\ \Delta_i B &:= p^\delta(\bar{\lambda}^{i+}) - p^\delta(\bar{\lambda}) = \sum_{(**)} \prod_{s \in U} p^{\delta_s}(\bar{\lambda}) \prod_{s' \in V} (p^{\delta_{s'}}(\bar{\lambda}^{i+}) - p^{\delta_{s'}}(\bar{\lambda})),\end{aligned}$$

where the sum $(*)$ (resp. $(**)$) ranges over all pairs (U, V) of positive integer sets such that $U \cup V = \{1, 2, \dots, \ell(\nu)\}$ (resp. $U \cup V = \{1, 2, \dots, \ell(\delta)\}$), $U \cap V = \emptyset$ and $V \neq \emptyset$.

Finally, it follows from (2.9) and Theorem 2.1 that

$$\begin{aligned}& H(\bar{\lambda}) \bar{D} \left(\frac{p^\delta(\bar{\lambda}) \Phi^\nu(\bar{\lambda})}{H(\bar{\lambda})} \right) \\&= \frac{H(\bar{\lambda})}{H(\bar{\lambda}^{0+})} (p^\delta(\bar{\lambda}^{0+}) \Phi^\nu(\bar{\lambda}^{0+}) - p^\delta(\bar{\lambda}) \Phi^\nu(\bar{\lambda})) \\&\quad + 2 \sum_{i=1}^m \frac{H(\bar{\lambda})}{H(\bar{\lambda}^{i+})} (p^\delta(\bar{\lambda}^{i+}) \Phi^\nu(\bar{\lambda}^{i+}) - p^\delta(\bar{\lambda}) \Phi^\nu(\bar{\lambda})) \\&= \sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} \left(\binom{x_i}{2} - \binom{y_j}{2} \right)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} \left(\binom{x_i}{2} - \binom{x_j}{2} \right)} (p^\delta(\bar{\lambda}^{i+}) \Phi^\nu(\bar{\lambda}^{i+}) - p^\delta(\bar{\lambda}) \Phi^\nu(\bar{\lambda})) \\&= \sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} \left(\binom{x_i}{2} - \binom{y_j}{2} \right)}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} \left(\binom{x_i}{2} - \binom{x_j}{2} \right)} (A \cdot \Delta_i B + B \cdot \Delta_i A + \Delta_i A \cdot \Delta_i B).\end{aligned}$$

By Theorems 2.4 and 2.2, each of the above three terms could be written as a linear combination of some $p^\delta(\bar{\lambda}) \Phi^\nu(\bar{\lambda})$ satisfying $|\underline{\delta}| + \ell(\underline{\delta}) + |\underline{\nu}| \leq |\delta| + \ell(\delta) + |\nu| - 1$. Then the claim follows by induction on $|\delta| + \ell(\delta) + |\nu|$. \square

When $\bar{\mu} = \emptyset$, the summation (2.13) in Theorem 2.6 becomes

$$(2.14) \quad \sum_{|\bar{\lambda}|=n} \frac{2^{n-\ell(\bar{\lambda})} n!}{H(\bar{\lambda})^2} p^\delta(\bar{\lambda})$$

or

$$(2.15) \quad 2^n n! \sum_{|\bar{\lambda}\bar{\lambda}|=2n} \frac{1}{H(\bar{\lambda}\bar{\lambda})} \Psi^\delta(\{h^2 : h \in \mathcal{H}(\bar{\lambda}\bar{\lambda})\}, \emptyset)$$

by (2.1). The above summation is a polynomial in n . Consequently, Theorem 1.2 is true when $t = 1$ and $F_2 = 1$. Other specializations are listed as follows.

Theorem 2.7. *Let $\bar{\mu}$ be a given strict partition. Then,*

$$(2.16) \quad \sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{|\bar{\lambda}|-\ell(\bar{\lambda})-|\bar{\mu}|+\ell(\bar{\mu})} f_{\bar{\lambda}/\bar{\mu}} H_{\bar{\mu}}}{H_{\bar{\lambda}}} (p^1(\bar{\lambda}) - p^1(\bar{\mu})) = 12 \binom{n}{2} + (12|\bar{\mu}| + 5)n.$$

Let $\bar{\mu} = \emptyset$. We obtain

$$(2.17) \quad 2^n n! \sum_{|\bar{\lambda}\bar{\lambda}|=2n} \frac{1}{H(\bar{\lambda}\bar{\lambda})} \sum_{h \in \mathcal{H}(\bar{\lambda}\bar{\lambda})} h^2 = 12 \binom{n}{2} + 5n.$$

Proof. We have

$$\begin{aligned} p^1(\bar{\lambda}^{0+}) - p^1(\bar{\lambda}) &= 2 \sum_{j=1}^m (x_j^2 + (x_j - 1)^2) - 2 \sum_{j=1}^m (y_j^2 + (y_j - 1)^2) + 2^2 + 1 \\ &= \eta_0(\bar{\lambda}) = 8|\bar{\lambda}| + 5 \quad [\text{ if } i = 0 \text{ and } \bar{\lambda}_{\ell(\bar{\lambda})} \geq 2] \end{aligned}$$

and

$$\begin{aligned} &p^1(\bar{\lambda}^{i+}) - p^1(\bar{\lambda}) \\ &= 2 \sum_{j=1}^m ((x_i - x_j)^2 + (x_i + x_j - 1)^2) - 2 \sum_{j=1}^m ((x_i - y_j)^2 + (x_i + y_j - 1)^2) \\ &\quad + 2x_i^2 + 2(2x_i - 2)^2 + 2 - 2(2x_i - 1)^2 + (2^2 - 1)(x_i^2 - (x_i - 1)^2) \\ &= 4 \binom{x_i}{2} + 8|\bar{\lambda}| + 5 \quad [\text{ if } 1 \leq i \leq m]. \end{aligned}$$

So that

$$\begin{aligned} H_{\bar{\lambda}} D \left(\frac{p^1(\bar{\lambda})}{H_{\bar{\lambda}}} \right) &= \sum_{0 \leq i \leq m} \frac{\prod_{1 \leq j \leq m} ((\binom{x_i}{2}) - (\binom{y_j}{2}))}{\prod_{\substack{0 \leq j \leq m \\ j \neq i}} ((\binom{x_i}{2}) - (\binom{x_j}{2}))} (4 \binom{x_i}{2} + 8|\bar{\lambda}| + 5) \\ &= 4\Phi^1(\bar{\lambda}) + 8|\bar{\lambda}| + 5 \\ &= 12|\bar{\lambda}| + 5. \end{aligned}$$

Therefore we have

$$\begin{aligned} H_{\bar{\lambda}} D^2 \left(\frac{p^1(\bar{\lambda})}{H_{\bar{\lambda}}} \right) &= 12, \\ H_{\bar{\lambda}} D^3 \left(\frac{p^1(\bar{\lambda})}{H_{\bar{\lambda}}} \right) &= 0. \end{aligned}$$

Identity (2.16) follows from Theorem 2.5. By (2.1), we derive (2.17). \square

Recall the following results obtained in [11] involving the contents of strict partitions.

Theorem 2.8. *Suppose that Q is a given symmetric function, and $\bar{\mu}$ is a given strict partition. Then*

$$\sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{|\bar{\lambda}|-|\bar{\mu}|-\ell(\bar{\lambda})+\ell(\bar{\mu})} f_{\bar{\lambda}/\bar{\mu}} Q \left(\binom{c}{2} : c \in \mathcal{C}(\bar{\lambda}) \right)}{H(\bar{\lambda})}$$

is a polynomial in n .

Theorem 2.9. *Suppose that k is a given nonnegative integer. Then*

$$\sum_{|\bar{\lambda}|=n} \frac{2^{|\bar{\lambda}|-\ell(\bar{\lambda})} f_{\bar{\lambda}}}{H(\bar{\lambda})} \sum_{c \in \mathcal{C}(\bar{\lambda})} \binom{c+k-1}{2k} = \frac{2^k}{(k+1)!} \binom{n}{k+1}.$$

Theorem 2.10. *Let $\bar{\mu}$ be a strict partition. Then,*

$$(2.18) \quad \sum_{|\bar{\lambda}/\bar{\mu}|=n} \frac{2^{|\bar{\lambda}|-\ell(\bar{\lambda})-|\bar{\mu}|+\ell(\bar{\mu})} f_{\bar{\lambda}/\bar{\mu}} H_{\bar{\mu}}}{H(\bar{\lambda})} \left(\sum_{c \in \mathcal{C}(\bar{\lambda})} \binom{c}{2} - \sum_{c \in \mathcal{C}(\bar{\mu})} \binom{c}{2} \right) = \binom{n}{2} + n|\bar{\mu}|.$$

The above results can be interpreted in terms of doubled distinct partitions. In particular, we obtain Theorem 1.2 when $t = 1$ and $F_1 = 1$.

Theorem 2.11. *For each usual partition δ , the summation*

$$(2.19) \quad 2^n n! \sum_{|\bar{\lambda}\bar{\lambda}|=2n} \frac{1}{H(\bar{\lambda}\bar{\lambda})} \Psi^\delta(\mathcal{C}(\bar{\lambda}\bar{\lambda}), \emptyset)$$

is a polynomial in n .

Proof. Since $c + (1 - c) = 1$ and $c(1 - c) = -2\binom{c}{2}$, there exists some a_i such that $c^k + (1 - c)^k = \sum_{i=1}^s a_i \binom{c}{2}^i$. By (2.2), we obtain

$$\sum_{c \in \mathcal{C}(\bar{\lambda}\bar{\lambda})} c^k = \sum_{c \in \mathcal{C}(\bar{\lambda})} (c^k + (1 - c)^k) = \sum_{i=1}^s a_i \sum_{c \in \mathcal{C}(\bar{\lambda})} \binom{c}{2}^i.$$

The claim follows from Theorem 2.8. \square

The following results are corollaries of Theorems 2.9 and 2.10.

Theorem 2.12. *Suppose that k is a given nonnegative integer. Then,*

$$(2.20) \quad 2^n n! \sum_{|\bar{\lambda}\bar{\lambda}|=2n} \frac{1}{H(\bar{\lambda}\bar{\lambda})} \sum_{c \in \mathcal{C}(\bar{\lambda}\bar{\lambda})} \binom{c+k-1}{2k} = \frac{2^{k+1}}{(k+1)!} \binom{n}{k+1},$$

$$(2.21) \quad 2^n n! \sum_{|\bar{\lambda}\bar{\lambda}|=2n} \frac{1}{H(\bar{\lambda}\bar{\lambda})} \sum_{c \in \mathcal{C}(\bar{\lambda}\bar{\lambda})} c^2 = 4 \binom{n}{2} + \binom{n}{1}.$$

3. THE LITTLEWOOD DECOMPOSITION AND CORNERS OF USUAL PARTITIONS

In this section we recall some basic definitions and properties for usual partitions (see [9], [15, p.12], [25, p.468], [12, p.75], [5]). Let \mathcal{W} be the set of bi-infinite binary sequences beginning with infinitely many 0's and ending with infinitely many 1's. Each element w of \mathcal{W} can be represented by $(a'_i)_i = \cdots a'_{-3}a'_{-2}a'_{-1}a'_0a'_1a'_2a'_3 \cdots$. However, the representation is not unique, since for any fixed integer k the sequence $(a'_{i+k})_i$ also represents w . The *canonical representation* of w is the unique sequence $(a_i)_i = \cdots a_{-3}a_{-2}a_{-1}a_0a_1a_2a_3 \cdots$ such that

$$\#\{i \leq -1, a_i = 1\} = \#\{i \geq 0, a_i = 0\}.$$

It will be further denoted by $\cdots a_{-3}a_{-2}a_{-1}.a_0a_1a_2a_3 \cdots$ with a dot symbol inserted between the letters a_{-1} and a_0 . There is a natural one-to-one correspondence between \mathcal{P} and \mathcal{W} (see, e.g. [25, p.468], [1] for more details). Let λ be a partition. We encode each horizontal edge of λ by 1 and each vertical edge by 0. Reading these (0,1)-encodings from top to bottom and from left to right yields a binary word u . By adding infinitely many 0's to the left and infinitely many 1's to the right of u we get an element $w = \cdots 000u111 \cdots \in \mathcal{W}$. Clearly, the map $\lambda \mapsto w$ is a one-to-one correspondence between \mathcal{P} and \mathcal{W} . For example, take $\lambda = (6, 3, 3, 1)$. Then $u = 0100110001$, so that $w = (a_i)_i = \cdots 1110100.110001000 \cdots$ (see Figure 5).

For a usual partition λ , the *outer corners* (see [10, 2]) are the boxes which can be removed to get a new partition. Let $(\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)$ be the coordinates of outer corners such that $\alpha_1 > \alpha_2 > \dots > \alpha_m$. Let $y_j = \beta_j - \alpha_j$ be the contents of outer corners for $1 \leq j \leq m$. We set $\alpha_{m+1} = \beta_0 = 0$ and call $(\alpha_1, \beta_0), (\alpha_2, \beta_1), \dots, (\alpha_{m+1}, \beta_m)$ the *inner corners* of λ . Let $x_i = \beta_i - \alpha_{i+1}$ be the contents of inner corners for $0 \leq i \leq m$ (see Figure 6). It is easy to verify that x_i and y_j satisfy the following relation:

$$(3.1) \quad x_0 < y_1 < x_1 < y_2 < x_2 < \dots < y_m < x_m.$$

Define (see [10])

$$(3.2) \quad \Psi^\nu(\lambda) := \Psi^\nu(\{x_i\}, \{y_j\})$$

for each usual partition ν .

Lemma 3.1. *Suppose that λ is a partition whose set of contents of inner corners and set of contents of outer corners are $X(\lambda) = \{x_0, x_1, \dots, x_m\}$ and $Y(\lambda) = \{y_1, y_2, \dots, y_m\}$ respectively. Let $\lambda^{i+} = \lambda \cup \{\square_i\}$ where $c_{\square_i} = x_i$. Then we have*

$$X(\lambda^{i+}) \cup \{x_i, x_i\} \cup Y(\lambda) = X(\lambda) \cup \{x_i + 1, x_i - 1\} \cup Y(\lambda^{i+}).$$

Proof. Four cases are to be considered. (i) If $\beta_i + 1 < \beta_{i+1}$ and $\alpha_{i+1} + 1 < \alpha_i$. Then, the contents of inner corners and outer corners of λ^{i+} are $X \cup \{x_i - 1, x_i + 1\} \setminus \{x_i\}$ and $Y \cup \{x_i\}$ respectively. (ii) If $\beta_i + 1 = \beta_{i+1}$ and $\alpha_{i+1} + 1 < \alpha_i$, so that $y_{i+1} = x_i + 1$. Hence the contents of inner corners and outer corners of λ^{i+} are $X \cup \{x_i - 1\} \setminus \{x_i\}$ and $Y \cup \{x_i\} \setminus \{x_i + 1\}$ respectively. (iii) If $\beta_i + 1 < \beta_{i+1}$ and $\alpha_{i+1} + 1 = \alpha_i$, so that $y_i = x_i - 1$. Then the contents of inner corners and outer corners of λ^{i+} are $X \cup \{x_i + 1\} \setminus \{x_i\}$ and $Y \cup \{x_i\} \setminus \{x_i - 1\}$ respectively. (iv) If $\beta_i + 1 = \beta_{i+1}$ and $\alpha_{i+1} + 1 = \alpha_i$. Then $y_i + 1 = x_i = y_{i+1} - 1$. The contents of inner corners and outer corners of λ^{i+} are $X \setminus \{x_i\}$ and $Y \cup \{x_i\} \setminus \{x_i - 1, x_i + 1\}$ respectively. The claim is proved. \square

The corners of the strict partition $\bar{\lambda}$ and the doubled distinct partition $\bar{\lambda}\bar{\lambda}$ are closely related.

Lemma 3.2. *Suppose that $\bar{\lambda}$ is a strict partition whose set of contents of inner corners and set of contents of outer corners are $X(\bar{\lambda}) = \{x_0, x_1, \dots, x_m\}$ and $Y(\bar{\lambda}) = \{y_1, y_2, \dots, y_m\}$ respectively. Then,*

$$X(\bar{\lambda}\bar{\lambda}) \cup \{y_1, 1 - y_1, \dots, y_m, 1 - y_m\} = Y(\bar{\lambda}\bar{\lambda}) \cup \{0, x_1, 1 - x_1, \dots, x_m, 1 - x_m\}.$$

Proof. Two cases are to be considered. (i) If $y_1 = 1$, the contents of inner corners and outer corners of $\bar{\lambda}\bar{\lambda}$ are $X(\bar{\lambda}\bar{\lambda}) = \{x_1, 1 - x_1, \dots, x_m, 1 - x_m\}$ and $Y(\bar{\lambda}\bar{\lambda}) = \{1, y_2, 1 - y_2, \dots, y_m, 1 - y_m\}$ respectively. (ii) If $y_1 \geq 2$, the contents of inner corners and outer corners of $\bar{\lambda}\bar{\lambda}$ are $X(\bar{\lambda}\bar{\lambda}) = \{0, x_1, 1 - x_1, \dots, x_m, 1 - x_m\}$ and $Y(\bar{\lambda}\bar{\lambda}) = \{y_1, 1 - y_1, \dots, y_m, 1 - y_m\}$ respectively. This achieves the proof of Lemma 3.2. \square

4. THE t -DIFFERENCE OPERATORS FOR DOUBLED DISTINCT PARTITIONS

Let $t = 2t' + 1$ be an odd positive integer. For each strict partition $\bar{\lambda}$, the doubled distinct partition associated with $\bar{\lambda}$ is denoted by $\lambda = \bar{\lambda}\bar{\lambda}$. The Littlewood decomposition maps $\bar{\lambda}\bar{\lambda}$ to

$$(\lambda_{t\text{-core}}; \lambda^0, \lambda^1, \dots, \lambda^{2t'}) \in \mathcal{DD}_{t\text{-core}} \times \mathcal{DD} \times \mathcal{P}^{2t'}$$

where λ^i is the conjugate partition of λ^{t-i} for $1 \leq i \leq t'$. For convenience we say that the *Littlewood decomposition* maps the strict partition $\bar{\lambda}$ to

$$(4.1) \quad \bar{\lambda} \mapsto (\bar{\lambda}_{t\text{-core}}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{t'}),$$

where $\bar{\lambda}_{t\text{-core}}$ and $\bar{\lambda}^0$ are determined by $\lambda_{t\text{-core}} = \bar{\lambda}_{t\text{-core}} \bar{\lambda}_{t\text{-core}}$ and $\lambda^0 = \bar{\lambda}^0 \bar{\lambda}^0$. Since the map (4.1) is bijective, we always write

$$\lambda = (\bar{\lambda}_{t\text{-core}}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{t'}).$$

Let $\lambda = \bar{\lambda} \bar{\lambda} = (\bar{\lambda}_{t\text{-core}}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{t'})$ and $\mu = \bar{\mu} \bar{\mu} = (\bar{\mu}_{t\text{-core}}; \bar{\mu}^0, \mu^1, \dots, \mu^{t'})$ be two doubled distinct partitions. If $\lambda_{t\text{-core}} = \mu_{t\text{-core}}$, $\bar{\lambda}^0 \supset \bar{\mu}^0$ and $\lambda^i \supset \mu^i$ for $1 \leq i \leq t'$, we write $\lambda \geq_t \mu$ and define

$$(4.2) \quad F_{\mu/\mu} := 1 \quad \text{and} \quad F_{\lambda/\mu} := \sum_{\substack{\lambda \geq_t \lambda^- \geq_t \mu \\ |\lambda/\lambda^-| = 2t}} F_{\lambda^-/\mu} \quad (\text{for } \lambda \neq \mu).$$

In fact, $F_{\lambda/\mu}$ is the number of vectors $(P_0, P_1, \dots, P_{t'})$ such that

- (1) P_0 is a skew shifted Young tableau of shape $\bar{\lambda}^0/\bar{\mu}^0$,
- (2) P_i ($1 \leq i \leq t'$) is a skew Young tableau of shape λ^i/μ^i ,
- (3) the union of entries in $P_0, P_1, \dots, P_{t'}$ are $1, 2, \dots, |\bar{\lambda}^0/\bar{\mu}^0| + \sum_{i=1}^{t'} |\lambda^i/\mu^i|$.

Hence,

$$F_{\lambda/\mu} = \left(|\bar{\lambda}^0/\bar{\mu}^0| + \sum_{i=1}^{t'} |\lambda^i/\mu^i| \right) f_{\bar{\lambda}^0/\bar{\mu}^0} \prod_{i=1}^{t'} f_{\lambda^i/\mu^i}.$$

We set

$$(4.3) \quad F_{\lambda} := F_{\lambda/\lambda_{t\text{-core}}} = \left(|\bar{\lambda}^0| + \sum_{i=1}^{t'} |\lambda^i| \right) f_{\bar{\lambda}^0} \prod_{i=1}^{t'} f_{\lambda^i} = \frac{n!}{H(\bar{\lambda}^0) \prod_{i=1}^{t'} H(\lambda^i)}$$

and

$$G_{\lambda} := \frac{2^{n-\ell(\bar{\lambda}^0)}}{t^n H(\bar{\lambda}^0) \prod_{i=1}^{t'} H(\lambda^i)} = \frac{2^{n-\ell(\bar{\lambda}^0)} F_{\lambda}}{t^n n!},$$

where $n = |\bar{\lambda}^0| + \sum_{i=1}^{t'} |\lambda^i|$.

When $t = 1$, we have $t' = 0$, thus $F_{\lambda} = f_{\bar{\lambda}^0}$ and $G_{\lambda} = 2^{n-\ell(\bar{\lambda}^0)}/H(\bar{\lambda}^0)$. Also, when λ is a t -core doubled distinct partition, we have $F_{\lambda} = G_{\lambda} = 1$.

4.1. t -difference operators. Let g be a function of doubled distinct partitions and λ be a doubled distinct partition. The t -difference operator D_t for doubled distinct partitions is defined by

$$(4.4) \quad D_t g(\lambda) = \sum_{\substack{\lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda| = 2t}} g(\lambda^+) - g(\lambda).$$

The higher-order t -difference operators D_t^k are defined by induction:

$$D_t^0 g := g \quad \text{and} \quad D_t^k g := D_t(D_t^{k-1} g) \quad (k \geq 1).$$

Lemma 4.1. *Let λ be a doubled distinct partition. Then,*

$$D_t(G_{\lambda}) = 0.$$

In other words,

$$G_\lambda = \sum_{\substack{\lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda|=2t}} G_{\lambda^+}.$$

Proof. Write $\lambda = (\bar{\lambda}_{t\text{-core}}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{t'})$. By Theorem 3.3 in [11] we obtain

$$\sum_{|(\bar{\lambda}^0)^+/\bar{\lambda}^0|=1} \frac{G_{(\bar{\lambda}_{t\text{-core}}; (\bar{\lambda}^0)^+, \lambda^1, \dots, \lambda^{t'})}}{G_\lambda} = \sum_{|(\bar{\lambda}^0)^+/\bar{\lambda}^0|=1} \frac{2^{1+\ell(\bar{\lambda}^0)-\ell((\bar{\lambda}^0)^+)} H(\bar{\lambda}^0)}{tH((\bar{\lambda}^0)^+)} = \frac{1}{t}.$$

For $1 \leq i \leq t'$ we have

$$\sum_{|(\lambda^i)^+/\lambda^i|=1} \frac{G_{(\bar{\lambda}_{t\text{-core}}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{i-1}, (\lambda^i)^+, \lambda^{i+1}, \dots, \lambda^{t'})}}{G_\lambda} = \sum_{|(\lambda^i)^+/\lambda^i|=1} \frac{2H(\lambda^i)}{tH((\lambda^i)^+)} = \frac{2}{t}$$

by Lemma 2.2 in [10]. Summing the above equalities, we get

$$\sum_{\substack{\lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda|=2t}} \frac{G_{\lambda^+}}{G_\lambda} = \frac{1}{t} + \sum_{i=1}^{t'} \frac{2}{t} = 1. \quad \square$$

Lemma 4.2. Suppose that μ is a given doubled distinct partition and g is a function of doubled distinct partitions. For every $n \in \mathbb{N}$, let

$$P(\mu, g; n) := \sum_{\substack{\lambda \in \mathcal{DD}, \lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} F_{\lambda/\mu} g(\lambda).$$

Then

$$P(\mu, g; n+1) - P(\mu, g; n) = P(\mu, D_t g; n).$$

Proof. The proof is straightforward:

$$\begin{aligned} P(\mu, g; n+1) - P(\mu, g; n) &= \sum_{\substack{\nu \geq_t \mu \\ |\nu/\mu|=2(n+1)t}} F_{\nu/\mu} g(\nu) - \sum_{\substack{\lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} F_{\lambda/\mu} g(\lambda) \\ &= \sum_{\substack{\nu \geq_t \mu \\ |\nu/\mu|=2(n+1)t}} \sum_{\substack{\nu \geq_t \nu^- \geq_t \mu \\ |\nu/\nu^-|=2t}} F_{\nu^-/\mu} g(\nu) - \sum_{\substack{\lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} F_{\lambda/\mu} g(\lambda) \\ &= \sum_{\substack{\lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} F_{\lambda/\mu} \left(\sum_{\substack{\lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda|=2t}} g(\lambda^+) - g(\lambda) \right) \\ &= P(\mu, D_t g; n). \quad \square \end{aligned}$$

Example. Let $g(\lambda) = G_\lambda$. Then $D_t g(\lambda) = 0$ by Lemma 4.1, which means that $P(\mu, D_t g; n) = 0$. Consequently, $P(\mu, G_\lambda; n+1) = P(\mu, G_\lambda; n) = \dots = P(\mu, G_\lambda; 0) = G_\mu$, or

$$(4.5) \quad \sum_{\substack{\lambda \in \mathcal{DD}, \lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} F_{\lambda/\mu} G_\lambda = G_\mu.$$

When μ is a t -core doubled distinct partition, the above identity becomes

$$\sum_{\substack{\lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} \frac{n!}{H(\bar{\lambda}^0) \prod_{i=1}^{t'} H(\lambda^i)} \times \frac{2^{n-\ell(\bar{\lambda}^0)}}{t^n H(\bar{\lambda}^0) \prod_{i=1}^{t'} H(\lambda^i)} = G_\mu,$$

or

$$\sum_{\substack{\lambda \in \mathcal{DD}, \\ \lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} \frac{(2t)^n n!}{H_t(\lambda)} = 1,$$

which implies (1.1).

Theorem 4.3. *Let g be a function of doubled distinct partitions and μ be a given doubled distinct partition. Then,*

$$(4.6) \quad P(\mu, g; n) = \sum_{\substack{\lambda \in \mathcal{DD}, \\ \lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} F_{\lambda/\mu} g(\lambda) = \sum_{k=0}^n \binom{n}{k} D_t^k g(\mu)$$

and

$$(4.7) \quad D_t^n g(\mu) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} P(\mu, g; k).$$

In particular, if there exists some positive integer r such that $D_t^r g(\lambda) = 0$ for every doubled distinct partition $\lambda \geq_t \mu$, then $P(\mu, g; n)$ is a polynomial in n with degree at most $r - 1$.

Proof. Identity (4.6) is proved by induction. The case $n = 0$ is obvious. Assume that (4.6) is true for some nonnegative integer n . By Lemma 4.2 we obtain

$$\begin{aligned} P(\mu, g; n+1) &= P(\mu, g; n) + P(\mu, D_t g; n) \\ &= \sum_{k=0}^n \binom{n}{k} D_t^k g(\mu) + \sum_{k=0}^n \binom{n}{k} D_t^{k+1} g(\mu) \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} D_t^k g(\mu). \end{aligned}$$

Identity (4.7) follows from the famous Möbius inversion formula [22]. \square

4.2. μ -admissible functions of doubled distinct partitions. Let $\mu = \bar{\mu}\bar{\mu}$ be a t -core doubled distinct partition. A function g of doubled distinct partitions is called μ -admissible, if for each given $1 \leq i \leq t'$ (resp. $i = 0$), $g(\lambda^+) - g(\lambda)$ is a polynomial in c_{\square_i} (resp. $\binom{c_{\square_0}}{2}$) for every pair of partitions

$$\lambda = (\bar{\mu}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{t'})$$

and

$$\begin{aligned} \lambda^+ &= (\bar{\mu}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{i-1}, \lambda^i \cup \square_i, \lambda^{i+1}, \dots, \lambda^{t'}) \\ &\text{(resp. } \lambda^+ = (\bar{\mu}; \bar{\lambda}^0 \cup \square_0, \lambda^1, \dots, \lambda^{t'}), \end{aligned}$$

whose coefficients are of form

$$\sum K(\mu, i; \tau^0, \tau^1, \dots, \tau^{t'}) \Phi^{\tau^0}(\bar{\lambda}^0) \prod_{j=1}^{t'} \Psi^{\tau^j}(\lambda^j),$$

where the summation is taken over the set of $(t' + 1)$ -tuple of usual partitions $(\tau^0, \tau^1, \dots, \tau^{t'})$ and K is a function.

Lemma 4.4. *Let μ be a t -core doubled distinct partition. Then, the two functions of doubled distinct partitions $\sum_{h \in \mathcal{H}(\lambda)} h^{2r}$ and $\sum_{c \in \mathcal{C}(\lambda)} c^r$ are μ -admissible for any nonnegative integer r .*

To prove Lemma 4.4, we recall some results on the multisets of hook lengths and contents, obtained in [3]. Suppose that a given t -core partition μ has 01-sequence $w(\mu) = (a_{\mu,j})_{j \in \mathbb{Z}}$. For $0 \leq i \leq t-1$ we define [3]

$$b_i := b_i(\mu) = \min\{j \in \mathbb{Z} : j \equiv i \pmod{t}, a_{\mu,j} = 1\}.$$

Lemma 4.5 (Lemma 5.3 of [3]). *Let λ be a partition and $(\lambda_{t\text{-core}}; \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ be the image of the Littlewood decomposition of λ . Then,*

$$\mathcal{C}(\lambda) \setminus \mathcal{C}(\lambda_{t\text{-core}}) = \bigcup_{i=0}^{t-1} \{tc + b_i(\lambda_{t\text{-core}}) - j : 0 \leq j \leq t-1, c \in \mathcal{C}(\lambda^i)\}.$$

Lemma 4.6 (Lemma 5.4 of [3]). *Let $0 \leq i \leq t-1$, λ and λ^+ be two usual partitions whose images of the Littlewood decomposition are $(\lambda_{t\text{-core}}; \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ and*

$$(\lambda_{t\text{-core}}; \lambda^0, \lambda^1, \dots, \lambda^{i-1}, \lambda^i \cup \{\square_i\}, \lambda^{i+1}, \dots, \lambda^{t-1})$$

respectively. Write $b_j = b_j(\lambda_{t\text{-core}})$ ($0 \leq j \leq t-1$). Suppose r is a given integer, $1 \leq k \leq t-1$. Let $x_{j,s}$ ($0 \leq s \leq m_j$) be the contents of inner corners of λ^j and $y_{j,s}$ ($1 \leq s \leq m_j$) be the contents of outer corners of λ^j for $0 \leq j \leq t-1$. We have

$$\sum_{\substack{\square \in \lambda^+ \\ h_{\square} \equiv 0 \pmod{t}}} h_{\square}^{2r} - \sum_{\substack{\square \in \lambda \\ h_{\square} \equiv 0 \pmod{t}}} h_{\square}^{2r} = t^{2r} + \sum_{0 \leq s \leq m_i} (t(c_{\square_i} - x_{i,s}))^{2r} - \sum_{1 \leq s \leq m_i} (t(c_{\square_i} - y_{i,s}))^{2r}$$

and

$$\begin{aligned} & \sum_{\substack{\square \in \lambda^+ \\ h_{\square} \equiv k \pmod{t}}} h_{\square}^{2r} + \sum_{\substack{\square \in \lambda^+ \\ h_{\square} \equiv t-k \pmod{t}}} h_{\square}^{2r} - \sum_{\substack{\square \in \lambda \\ h_{\square} \equiv k \pmod{t}}} h_{\square}^{2r} - \sum_{\substack{\square \in \lambda \\ h_{\square} \equiv t-k \pmod{t}}} h_{\square}^{2r} \\ &= \sum_{0 \leq s \leq m_{i'}} (tc_{\square_i} + b_i - tx_{i',s} - b_{i'})^{2r} - \sum_{1 \leq s \leq m_{i'}} (tc_{\square_i} + b_i - ty_{i',s} - b_{i'})^{2r} \\ &+ \sum_{0 \leq s \leq m_{i''}} (tc_{\square_i} + b_i - tx_{i'',s} - b_{i''})^{2r} - \sum_{1 \leq s \leq m_{i''}} (tc_{\square_i} + b_i - ty_{i'',s} - b_{i''})^{2r} \end{aligned}$$

where $0 \leq i', i'' \leq t-1$ satisfy $i' \equiv i+k \pmod{t}$ and $i'' \equiv i-k \pmod{t}$. Furthermore,

$$\begin{aligned} \sum_{\square \in \lambda^+} h_{\square}^{2r} - \sum_{\square \in \lambda} h_{\square}^{2r} &= t^{2r} + \sum_{j=0}^{t-1} \left(\sum_{0 \leq s \leq m_j} (tc_{\square_j} + b_i - tx_{j,s} - b_j)^{2r} \right. \\ &\quad \left. - \sum_{1 \leq s \leq m_j} (tc_{\square_j} + b_i - ty_{j,s} - b_j)^{2r} \right). \end{aligned}$$

For the doubled distinct partition λ whose image under Littlewood decomposition is $(\lambda_{t\text{-core}}; \lambda^0, \lambda^1, \dots, \lambda^{t-1})$ where $\lambda^0 = \bar{\lambda}^0 \bar{\lambda}^0$, let $x_{0,s}$ ($0 \leq s \leq m_0$) be the contents of inner corners of $\bar{\lambda}^0$ and $y_{0,s}$ ($1 \leq s \leq m_0$) be the contents of outer

corners of $\bar{\lambda}^0$. Let $x_{i,s}$ ($0 \leq s \leq m_i$) be the contents of inner corners of λ^i and $y_{i,s}$ ($1 \leq s \leq m_i$) be the contents of outer corners of λ^i for $1 \leq i \leq t-1$. Then $x_{i,s} = -x_{t-i, m_i-s}$ and $y_{i,s} = -y_{t-i, m_i+1-s}$ since λ^i and λ^{t-i} are conjugate to each other for $1 \leq i \leq t-1$.

Proof of Lemma 4.4. Let $\lambda = (\bar{\lambda}_{t\text{-core}}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{t'})$ be a doubled distinct partition and $b_j = b_j(\bar{\lambda}_{t\text{-core}} \bar{\lambda}_{t\text{-core}})$ for $0 \leq j \leq t-1$. The following statements are consequences of Lemma 4.5.

(C1) Let $1 \leq i \leq t'$ and $\lambda^+ = (\bar{\lambda}_{t\text{-core}}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{i-1}, (\lambda^i)^+, \lambda^{i+1}, \dots, \lambda^{t'})$ be a doubled distinct partition such that $(\lambda^i)^+ = \lambda^i \cup \square_i$, we have

$$\mathcal{C}(\lambda^+) \setminus \mathcal{C}(\lambda) = \{tc_{\square_i} + b_i - j : 0 \leq j \leq t-1\} \cup \{-tc_{\square_i} + b_{t-i} - j : 0 \leq j \leq t-1\}.$$

(C2) Let $\lambda^+ = (\bar{\lambda}_{t\text{-core}}; (\bar{\lambda}^0)^+, \lambda^1, \dots, \lambda^{t'})$ be a doubled distinct partition such that $(\bar{\lambda}^0)^+ = \bar{\lambda}^0 \cup \square_0$, we have

$$\mathcal{C}(\lambda^+) \setminus \mathcal{C}(\lambda) = \{tc_{\square_0} + b_0 - j : 0 \leq j \leq t-1\} \cup \{t(1 - c_{\square_0}) + b_0 - j : 0 \leq j \leq t-1\}.$$

Hence, $\sum_{c \in \mathcal{C}(\lambda)} c^r$ is μ -admissible for any nonnegative integer r .

On the other hand, we obtain the following results by Lemma 4.6.

(H1) Let $1 \leq i \leq t'$ and $\lambda^+ = (\bar{\lambda}_{t\text{-core}}; \bar{\lambda}^0, \lambda^1, \dots, \lambda^{i-1}, (\lambda^i)^+, \lambda^{i+1}, \dots, \lambda^{t'})$ be a doubled distinct partition such that $(\lambda^i)^+ = \lambda^i \cup \square_i$, we have

$$\begin{aligned} & \sum_{\square \in \lambda^+} h_{\square}^{2r} - \sum_{\square \in \lambda} h_{\square}^{2r} \\ &= t^{2r} + (tc_{\square_i} + b_i - b_0)^{2r} \\ &+ \sum_{j=1}^{t-1} \left(\sum_{0 \leq s \leq m_j} (tc_{\square_i} + b_i - tx_{j,s} - b_j)^{2r} - \sum_{1 \leq s \leq m_j} (tc_{\square_i} + b_i - ty_{j,s} - b_j)^{2r} \right) \\ &+ \sum_{1 \leq s \leq m_j} (tc_{\square_i} + b_i - tx_{0,s} - b_0)^{2r} - \sum_{1 \leq s \leq m_j} (tc_{\square_i} + b_i - ty_{0,s} - b_0)^{2r} \\ &+ \sum_{1 \leq s \leq m_j} (tc_{\square_i} + b_i - t(1 - x_{0,s}) - b_0)^{2r} - \sum_{1 \leq s \leq m_j} (tc_{\square_i} + b_i - t(1 - y_{0,s}) - b_0)^{2r} \\ &+ t^{2r} + (-tc_{\square_i} + b_{t-i} - b_0)^{2r} \\ &+ \sum_{j=1}^{t-1} \left(\sum_{0 \leq s \leq m_j} (-tc_{\square_i} + b_{t-i} - tx_{j,s} - b_j)^{2r} - \sum_{1 \leq s \leq m_j} (-tc_{\square_i} + b_{t-i} - ty_{j,s} - b_j)^{2r} \right) \\ &+ \sum_{1 \leq s \leq m_j} (-tc_{\square_i} + b_{t-i} - tx_{0,s} - b_0)^{2r} - \sum_{1 \leq s \leq m_j} (-tc_{\square_i} + b_{t-i} - ty_{0,s} - b_0)^{2r} \\ &+ \sum_{1 \leq s \leq m_j} (-tc_{\square_i} + b_{t-i} - t(1 - x_{0,s}) - b_0)^{2r} \\ &- \sum_{1 \leq s \leq m_j} (-tc_{\square_i} + b_{t-i} - t(1 - y_{0,s}) - b_0)^{2r} + (-tc_{\square_i} + b_{t-i} - t(c_{\square_i} + 1) - b_0)^{2r} \\ &+ (-tc_{\square_i} + b_{t-i} - t(c_{\square_i} - 1) - b_0)^{2r} - 2(-tc_{\square_i} + b_{t-i} - tc_{\square_i} - b_0)^{2r}. \end{aligned}$$

(H2) Let $\lambda^+ = (\bar{\lambda}_{t\text{-core}}; (\bar{\lambda}^0)^+, \lambda^1, \dots, \lambda^{t'})$ be a doubled distinct partition such that $(\bar{\lambda}^0)^+ = \bar{\lambda}^0 \cup \square_0$, we have

$$\begin{aligned}
& \sum_{\square \in \lambda^+} h_{\square}^{2r} - \sum_{\square \in \lambda} h_{\square}^{2r} \\
&= t^{2r} + (tc_{\square_0})^{2r} \\
&+ \sum_{j=1}^{t-1} \left(\sum_{0 \leq s \leq m_j} (tc_{\square_0} + b_0 - tx_{j,s} - b_j)^{2r} - \sum_{1 \leq s \leq m_j} (tc_{\square_0} + b_0 - ty_{j,s} - b_j)^{2r} \right) \\
&+ \sum_{1 \leq s \leq m_j} (tc_{\square_0} - tx_{0,s})^{2r} - \sum_{1 \leq s \leq m_j} (tc_{\square_0} - ty_{0,s})^{2r} \\
&+ \sum_{1 \leq s \leq m_j} (tc_{\square_0} - t(1 - x_{0,s}))^{2r} - \sum_{1 \leq s \leq m_j} (tc_{\square_0} - t(1 - y_{0,s}))^{2r} \\
&+ t^{2r} + (t - tc_{\square_0})^{2r} \\
&+ \sum_{j=1}^{t-1} \left(\sum_{0 \leq s \leq m_j} (t - tc_{\square_0} + b_0 - tx_{j,s} - b_j)^{2r} - \sum_{1 \leq s \leq m_j} (t - tc_{\square_0} + b_0 - ty_{j,s} - b_j)^{2r} \right) \\
&+ \sum_{1 \leq s \leq m_j} (t - tc_{\square_0} - tx_{0,s})^{2r} - \sum_{1 \leq s \leq m_j} (t - tc_{\square_0} - ty_{0,s})^{2r} \\
&+ \sum_{1 \leq s \leq m_j} (t - tc_{\square_0} - t(1 - x_{0,s}))^{2r} - \sum_{1 \leq s \leq m_j} (t - tc_{\square_0} - t(1 - y_{0,s}))^{2r} \\
&+ (t - tc_{\square_0} - t(c_{\square_0} + 1))^{2r} + (t - tc_{\square_0} - t(c_{\square_0} - 1))^{2r} \\
&- 2(t - tc_{\square_0} - tc_{\square_0})^{2r}.
\end{aligned}$$

Hence, $\sum_{h \in \mathcal{H}(\lambda)} h^{2r}$ is μ -admissible for any nonnegative integer r . \square

4.3. Main results for doubled distinct partitions. To prove the doubled distinct case of Theorem 1.2, we establish the following more general result.

Theorem 4.7. *Let $(\nu^0, \nu^1, \dots, \nu^{t'})$ be a $(t' + 1)$ -tuple of usual partitions, and α be a t -core doubled distinct partition. Suppose that g_1, g_2, \dots, g_v are α -admissible functions of doubled distinct partitions. Then, there exists some $r \in \mathbb{N}$ such that*

$$D_t^r \left(G_{\lambda} \prod_{u=1}^v g_u(\lambda) \Phi^{\nu^0}(\bar{\lambda}^0) \prod_{i=1}^{t'} \Psi^{\nu^i}(\lambda^i) \right) = 0$$

for every doubled distinct partition λ with $\lambda_{t\text{-core}} = \alpha$. Furthermore, let μ be a given doubled distinct partition. By Theorem 4.3,

$$(4.8) \quad \sum_{\substack{\lambda \in \mathcal{DD}, \lambda \geq_t \mu \\ |\lambda/\mu| = 2nt}} F_{\lambda/\mu} G_{\lambda} \prod_{u=1}^v g_u(\lambda)$$

is a polynomial in n .

Proof. We will prove this claim by induction. Let

$$A = \prod_{u=1}^v g_u(\lambda), \quad B = \prod_{i=1}^{t'} \Psi^{\nu^i}(\lambda^i), \quad C = \Phi^{\nu^0}(\bar{\lambda}^0),$$

$$\begin{aligned}
\Delta A &= \prod_{u=1}^v g_u(\rho) - \prod_{u=1}^v g_u(\lambda) = \sum_{(*)} \prod_{s \in U} g_s(\lambda) \prod_{s' \in V} (g_{s'}(\rho) - g_{s'}(\lambda)), \\
\Delta B &= \prod_{i=1}^{t'} \Psi^{\nu^i}(\rho^i) - \prod_{i=1}^{t'} \Psi^{\nu^i}(\lambda^i) \\
&= \sum_{(**)} \prod_{s \in U} \Psi^{\nu^s}(\lambda^i) \prod_{s' \in V} (\Psi^{\nu^{s'}}(\rho^i) - \Psi^{\nu^{s'}}(\lambda^i)), \\
\Delta C &= \Phi^{\nu^0}(\bar{\rho}^0) - \Phi^{\nu^0}(\bar{\lambda}^0),
\end{aligned}$$

where the sum $(*)$ (resp. $(**)$) ranges over all pairs (U, V) of positive integer sets such that $U \cup V = \{1, 2, \dots, v\}$ (resp. $U \cup V = \{1, 2, \dots, t'\}$), $U \cap V = \emptyset$ and $V \neq \emptyset$. We have

$$\begin{aligned}
& D_t \left(G_\lambda \prod_{u=1}^v g_u(\lambda) \Phi^{\nu^0}(\bar{\lambda}^0) \prod_{i=1}^{t'} \Psi^{\nu^i}(\lambda^i) \right) \\
&= G_\lambda \sum_{\substack{\rho \geq_t \lambda \\ |\rho/\lambda|=2t}} \frac{G_\rho}{G_\lambda} \left(\prod_{u=1}^v g_u(\rho) \Phi^{\nu^0}(\bar{\rho}^0) \prod_{i=1}^{t'} \Psi^{\nu^i}(\rho^i) \right. \\
&\quad \left. - \prod_{u=1}^v g_u(\lambda) \Phi^{\nu^0}(\bar{\lambda}^0) \prod_{i=1}^{t'} \Psi^{\nu^i}(\lambda^i) \right) \\
&= G_\lambda \sum_{\substack{\rho \geq_t \lambda \\ |\rho/\lambda|=2t}} \frac{G_\rho}{G_\lambda} (\Delta A \cdot B \cdot C + A \cdot \Delta B \cdot C + A \cdot B \cdot \Delta C \\
&\quad + A \cdot \Delta B \cdot \Delta C + \Delta A \cdot B \cdot \Delta C + \Delta A \cdot \Delta B \cdot C + \Delta A \cdot \Delta B \cdot \Delta C).
\end{aligned}$$

For the first term in the above summation, we obtain

$$\begin{aligned}
& G_\lambda \sum_{\substack{\rho \geq_t \lambda \\ |\rho/\lambda|=2t}} \frac{G_\rho}{G_\lambda} (\Delta A \cdot B \cdot C) \\
&= \frac{1}{t} G_\lambda \Phi^{\nu^0}(\bar{\lambda}^0) \prod_{i=1}^{t'} \Psi^{\nu^i}(\lambda^i) \sum_{0 \leq i \leq m_0} \frac{\prod_{1 \leq j \leq m_0} ((x_{0,i}) - (y_{0,j}))}{\prod_{\substack{0 \leq j \leq m_0 \\ j \neq i}} ((x_{0,i}) - (x_{0,j}))} \\
&\quad \times \sum_{(*)} \prod_{s \in U} g_s(\lambda) \prod_{s' \in V} (g_{s'}((\bar{\mu}; (\bar{\lambda}^0)^{i+}, \lambda^1, \dots, \lambda^{t'})) - g_{s'}(\lambda)) \\
&+ \frac{2}{t} G_\lambda \Phi^{\nu^0}(\bar{\lambda}^0) \prod_{i=1}^{t'} \Psi^{\nu^i}(\lambda^i) \sum_{k=1}^{t'} \sum_{0 \leq i \leq m_k} \frac{\prod_{1 \leq j \leq m_k} (x_{k,i} - y_{k,j})}{\prod_{\substack{0 \leq j \leq m_k \\ j \neq i}} (x_{k,i} - x_{k,j})} \\
&\quad \times \sum_{(*)} \prod_{s \in U} g_s(\lambda) \prod_{s' \in V} (g_{s'}((\bar{\mu}; \bar{\lambda}^0, \lambda^1, \dots, (\lambda^k)^{i+}, \dots, \lambda^{t'})) - g_{s'}(\lambda))
\end{aligned}$$

where $\mathcal{C}((\bar{\lambda}^0)^{i+}) \setminus \mathcal{C}(\bar{\lambda}^0) = x_{0,i}$ and $\mathcal{C}((\lambda^k)^{i+}) \setminus \mathcal{C}(\lambda^k) = x_{k,i}$. Since g_1, g_2, \dots, g_v are α -admissible functions and thanks to Lemma 2.4,

$$G_\lambda \sum_{\substack{\rho \geq_t \lambda \\ |\rho/\lambda|=2t}} \frac{G_\rho}{G_\lambda} (\Delta A \cdot B \cdot C)$$

could be written as a linear combination of some

$$G_\lambda \prod_{\underline{u}=1}^{\underline{v}} g_{\underline{u}}(\lambda) \Phi^{\underline{\nu}^0}(\bar{\lambda}^0) \prod_{i=1}^{t'} \Psi^{\underline{\nu}^i}(\lambda^i)$$

where either $\underline{v} < v$, or $\underline{v} = v$ and simultaneously

$$2|\underline{\nu}^0| + \sum_{i=1}^{t'} |\underline{\nu}^i| \leq 2|\nu^0| + \sum_{i=1}^{t'} |\nu^i| - 2.$$

In the other hand, we have similar results for other six terms by Lemmas 3.1, 2.4 and Theorem 2.2. Thus, Theorem 4.7 is proved by induction on $(v, 2|\nu^0| + \sum_{i=1}^{t'} |\nu^i|)$. \square

As an application of Theorem 4.7, we derive the doubled distinct case of Theorem 1.2 from Lemma 4.4. Actually, by a similar but more precise argument as in the proof of Lemma 4.4, we can show that

$$\sum_{\substack{\square \in \lambda \\ h_\square \equiv \pm j \pmod{t}}} h_\square^{2r} \quad \text{and} \quad \sum_{\substack{\square \in \lambda \\ c_\square \equiv j \pmod{t}}} c_\square^r$$

are μ -admissible for any t -core doubled distinct partition μ , any nonnegative integer r , and $0 \leq j \leq t-1$. By Theorem 4.7 we derive the following result.

Corollary 4.8. *Let $u', v', j_u, j_v, k_u, k_v$ be nonnegative integers and α be a given t -core doubled distinct partition. Then, there exists some $r \in \mathbb{N}$ such that*

$$D_t^r \left(G_\lambda \left(\prod_{u=1}^{u'} \sum_{\substack{\square \in \lambda \\ h_\square \equiv \pm j_u \pmod{t}}} h_\square^{2k_u} \right) \left(\prod_{v=1}^{v'} \sum_{\substack{\square \in \lambda \\ c_\square \equiv j_v \pmod{t}}} c_\square^{k_v} \right) \right) = 0$$

for every doubled distinct partition λ with $\lambda_{t\text{-core}} = \alpha$. Moreover, let μ be a given doubled distinct partition.

$$\sum_{\substack{\lambda \in \mathcal{DD}, \lambda \geq_t \mu \\ |\lambda/\mu|=2nt}} F_{\lambda/\mu} G_\lambda \left(\prod_{u=1}^{u'} \sum_{\substack{\square \in \lambda \\ h_\square \equiv \pm j_u \pmod{t}}} h_\square^{2k_u} \right) \left(\prod_{v=1}^{v'} \sum_{\substack{\square \in \lambda \\ c_\square \equiv j_v \pmod{t}}} c_\square^{k_v} \right)$$

is a polynomial in n of degree at most $\sum_{u=1}^{u'} (k_u + 1) + \sum_{v=1}^{v'} \frac{k_v + 2}{2}$.

5. POLYNOMIALITY FOR SELF-CONJUGATE PARTITIONS

In this section we always set that $t = 2t'$ is an even integer. The set of all t -core self-conjugate partitions is denoted by $\mathcal{SC}_{t\text{-core}}$. Let λ be a self-conjugate partition. By [5], the Littlewood decomposition maps λ to

$$(\lambda_{t\text{-core}}; \lambda^0, \lambda^1, \dots, \lambda^{t-1}) \in \mathcal{SC}_{t\text{-core}} \times \mathcal{P}^{2t'}$$

where λ^i is the conjugate partition of λ^{t-1-i} for $0 \leq i \leq t' - 1$. For convenience, we always write

$$\lambda = (\lambda_{t\text{-core}}; \lambda^0, \dots, \lambda^{t'-1}).$$

Let $\lambda = (\lambda_{t\text{-core}}; \lambda^0, \dots, \lambda^{t'-1})$ and $\mu = (\mu_{t\text{-core}}; \mu^0, \dots, \mu^{t'-1})$ be two self-conjugate partitions. If $\lambda_{t\text{-core}} = \mu_{t\text{-core}}$ and $\lambda^i \supset \mu^i$ for $0 \leq i \leq t' - 1$, we write $\lambda \geq_t \mu$ and define

$$(5.1) \quad F_{\mu/\mu} := 1 \quad \text{and} \quad F_{\lambda/\mu} := \sum_{\substack{\lambda \geq_t \lambda^- \geq_t \mu \\ |\lambda/\lambda^-| = 2t}} F_{\lambda^-/\mu} \quad (\text{for } \lambda \neq \mu).$$

Then $F_{\lambda/\mu}$ is the number of vectors $(P_0, P_1, \dots, P_{t'-1})$ such that

- (1) P_i ($0 \leq i \leq t' - 1$) is a skew Young tableau of shape λ^i/μ^i ,
- (2) the union of entries in $P_0, P_1, \dots, P_{t'}$ are $1, 2, \dots, n = \sum_{i=0}^{t'-1} |\lambda^i/\mu^i|$.

Hence,

$$F_{\lambda/\mu} = \left(\sum_{i=0}^{t'-1} |\lambda^i/\mu^i| \right) \prod_{i=0}^{t'-1} f_{\lambda^i/\mu^i}.$$

We set

$$(5.2) \quad F_\lambda := F_{\lambda/\lambda_{t\text{-core}}} = \left(\sum_{i=0}^{t'-1} |\lambda^i| \right) \prod_{i=0}^{t'-1} f_{\lambda^i} = \frac{n!}{\prod_{i=0}^{t'-1} H(\lambda^i)}$$

and

$$G_\lambda := \frac{2^n}{t^n \prod_{i=0}^{t'-1} H(\lambda^i)} = \frac{2^n F_\lambda}{t^n n!}.$$

Let g be a function of self-conjugate partitions and λ be a self-conjugate partition. The t -difference operator D_t for self-conjugate partitions is defined by

$$(5.3) \quad D_t g(\lambda) = \sum_{\substack{\lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda| = 2t}} g(\lambda^+) - g(\lambda).$$

The higher-order t -difference operators D_t^k are defined by induction:

$$D_t^0 g := g \quad \text{and} \quad D_t^k g := D_t(D_t^{k-1} g) \quad (k \geq 1).$$

Lemma 5.1. *Suppose that λ is a self-conjugate partition. Then $D_t(G_\lambda) = 0$. In other words,*

$$(5.4) \quad G_\lambda = \sum_{\substack{\lambda^+ \geq_t \lambda \\ |\lambda^+/\lambda| = 2t}} G_{\lambda^+}.$$

Proof. Write $\lambda = (\lambda_{t\text{-core}}; \lambda^0, \dots, \lambda^{t'-1})$. For $0 \leq i \leq t' - 1$ we obtain

$$\sum_{|(\lambda^i)^+/\lambda^i|=1} \frac{G_{(\lambda_{t\text{-core}}; \lambda^0, \dots, \lambda^{i-1}, (\lambda^i)^+, \lambda^{i+1}, \dots, \lambda^{t'-1})}}{G_\lambda} = \sum_{|(\lambda^i)^+/\lambda^i|=1} \frac{2H(\lambda^i)}{tH((\lambda^i)^+)} = \frac{2}{t}$$

by Lemma 2.2 in [10]. Summing the above equalities we prove (5.4). \square

By analogy with the results on doubled distinct partitions, we have the following theorems for self-conjugate partitions. Their proofs are omitted.

Lemma 5.2. *Suppose that μ is a given self-conjugate partition and g is a function of self-conjugate partitions. For every nonnegative integer n , let*

$$P(\mu, g; n) := \sum_{\substack{\lambda \in SC, \lambda \geq_t \mu \\ |\lambda/\mu| = 2nt}} F_{\lambda/\mu} g(\lambda).$$

Then

$$P(\mu, g; n+1) - P(\mu, g; n) = P(\mu, D_t g; n).$$

Example. Let $g(\lambda) = G_\lambda$. Then $D_t g(\lambda) = 0$ by Lemma 5.1, which means that

$$(5.5) \quad \sum_{\substack{\lambda \in SC, \lambda \geq_t \mu \\ |\lambda/\mu| = 2nt}} F_{\lambda/\mu} G_\lambda = G_\mu.$$

When $\mu = \emptyset$, the above identity becomes

$$(5.6) \quad \sum_{\substack{\lambda \in SC, |\lambda| = 2nt \\ \lambda_{t\text{-core}} = \emptyset}} \frac{(2t)^n n!}{\prod_{h \in \mathcal{H}_t(\lambda)} h} = 1.$$

Theorem 5.3. *Let g be a function of self-conjugate partitions and μ be a given self-conjugate partition. Then,*

$$(5.7) \quad P(\mu, g; n) = \sum_{\substack{\lambda \in SC, \lambda \geq_t \mu \\ |\lambda/\mu| = 2nt}} F_{\lambda/\mu} g(\lambda) = \sum_{k=0}^n \binom{n}{k} D_t^k g(\mu)$$

and

$$(5.8) \quad D_t^n g(\mu) = \sum_{k=0}^n (-1)^{n+k} \binom{n}{k} P(\mu, g; k).$$

In particular, if there exists some positive integer r such that $D_t^r g(\lambda) = 0$ for every self-conjugate partition $\lambda \geq_t \mu$, then $P(\mu, g; n)$ is a polynomial in n of degree at most $r-1$.

Theorem 5.4. *Let $t = 2t'$ be a given integer, α be a given t -core self-conjugate partition, and $u', v', j_u, j'_v, k_u, k'_v$ be nonnegative integers. Then there exists some $r \in \mathbb{N}$ such that*

$$D_t^r \left(\prod_{u=1}^{u'} \sum_{\substack{\square \in \lambda \\ h_{\square} \equiv \pm j_u \pmod{t}}} h_{\square}^{2k_u} \right) \left(\prod_{v=1}^{v'} \sum_{\substack{\square \in \lambda \\ c_{\square} \equiv j'_v \pmod{t}}} c_{\square}^{k'_v} \right) = 0$$

for every self-conjugate partition λ with $\lambda_{t\text{-core}} = \alpha$. Furthermore, let μ be a given self-conjugate partition. Then by Theorem 5.3, we have

$$\sum_{\substack{\lambda \in SC, \lambda \geq_t \mu \\ |\lambda/\mu| = 2nt}} F_{\lambda/\mu} G_\lambda \left(\prod_{u=1}^{u'} \sum_{\substack{\square \in \lambda \\ h_{\square} \equiv \pm j_u \pmod{t}}} h_{\square}^{2k_u} \right) \left(\prod_{v=1}^{v'} \sum_{\substack{\square \in \lambda \\ c_{\square} \equiv j'_v \pmod{t}}} c_{\square}^{k'_v} \right)$$

is a polynomial in n of degree at most $\sum_{u=1}^{u'} (k_u + 1) + \sum_{v=1}^{v'} \frac{k'_v + 2}{2}$.

6. SQUARE CASES FOR DOUBLED DISTINCT AND SELF-CONJUGATE PARTITIONS

As described in Corollary 1.3, the polynomials mentioned in Corollary 4.8 and Theorem 5.4 have explicit expressions for square cases.

Proof of Corollary 1.3. (1) When λ is a doubled distinct partition with $|\lambda| = 2nt$ (t odd) and $\lambda_{t\text{-core}} = \emptyset$. By the proof of Lemma 4.4 we obtain

$$\begin{aligned}
\frac{1}{G_\lambda} D(G_\lambda(\sum_{\square \in \lambda} c_\square^2)) &= \frac{1}{t} \sum_{0 \leq i \leq m_0} \frac{\prod_{1 \leq j \leq m_0} ((\binom{x_{0,i}}{2} - (\binom{y_{0,j}}{2}))}{\prod_{\substack{0 \leq j \leq m_0 \\ j \neq i}} ((\binom{x_{0,i}}{2} - (\binom{x_{0,j}}{2}))} \\
&\quad \times \sum_{j=0}^{t-1} ((tx_{0,i} - j)^2 + (t - tx_{0,i} - j)^2) \\
&\quad + \frac{2}{t} \sum_{1 \leq k \leq t'} \sum_{0 \leq i \leq m_k} \frac{\prod_{1 \leq j \leq m_k} (x_{k,i} - y_{k,j})}{\prod_{\substack{0 \leq j \leq m_k \\ j \neq i}} (x_{k,i} - x_{k,j})} \\
&\quad \times \sum_{j=0}^{t-1} ((tx_{k,i} + k - j)^2 + (-tx_{k,i} + t - k - j)^2) \\
&= \frac{1}{t} \sum_{0 \leq i \leq m_0} \frac{\prod_{1 \leq j \leq m_0} ((\binom{x_{0,i}}{2} - (\binom{y_{0,j}}{2}))}{\prod_{\substack{0 \leq j \leq m_0 \\ j \neq i}} ((\binom{x_{0,i}}{2} - (\binom{x_{0,j}}{2}))} \\
&\quad \times (4t^3 \binom{x_{0,i}}{2} + t^3 - t^2(t-1) + \frac{(t-1)t(2t-1)}{3}) \\
&\quad + \frac{2}{t} \sum_{1 \leq k \leq t'} \sum_{0 \leq i \leq m_k} \frac{\prod_{1 \leq j \leq m_k} (x_{k,i} - y_{k,j})}{\prod_{\substack{0 \leq j \leq m_k \\ j \neq i}} (x_{k,i} - x_{k,j})} \\
&\quad \times (2t^3 x_{k,i}^2 + \sum_{j=0}^{t-1} (k-j)^2 + \sum_{j=0}^{t-1} (t-k-j)^2) \\
&= 2t|\lambda| + \frac{t(t^2+2)}{3},
\end{aligned}$$

therefore

$$\frac{1}{G_\lambda} D^2(G_\lambda(\sum_{\square \in \lambda} c_\square^2)) = 4t^2,$$

and

$$\frac{1}{G_\lambda} D^3(G_\lambda(\sum_{\square \in \lambda} c_\square^2)) = 0.$$

(2) When λ is a self-conjugate partition with $|\lambda| = 2nt$ (t even) and $\lambda_{t\text{-core}} = \emptyset$. Similarly as in (1) we have

$$\frac{1}{G_\lambda} D(G_\lambda(\sum_{\square \in \lambda} c_\square^2)) = 2t|\lambda| + \frac{t(t^2-1)}{3},$$

$$\frac{1}{G_\lambda} D^2(G_\lambda(\sum_{\square \in \lambda} c_\square^2)) = 4t^2,$$

and

$$\frac{1}{G_\lambda} D^3(G_\lambda(\sum_{\square \in \lambda} c_\square^2)) = 0.$$

Then identities (1.7) and (1.8) follows from Theorems 4.3 and 5.3. Notice that $\sum_{\square \in \lambda} h_\square^2 - \sum_{\square \in \lambda} c_\square^2 = |\lambda|^2$ (see [15]). Identities (1.5) and (1.6) are consequences of identities (1.7) and (1.8). \square

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